# Fake diffusions

Option pricing, fake Brownian motion and minimising the 1-variation

David Hobson
University of Warwick
www.warwick.ac.uk/go/dhobson

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# The problem

Given a family of marginals (increasing in convex order) construct a martingale with these marginals.

Fake diffusion variant:

Let $Y = (Y_t)_{0 \leq t \leq T}$ be a martingale on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F} = (\mathcal{F})_{t \geq 0})$.

Construct a martingale $X = (X_t)_{0 \leq t \leq T}$, different from $Y$, which has the same 1-marginals as $Y$. $X$ may live on a different probability space.

Optimality variant:

Construct a consistent process $X$ for which some functional on the paths is minimised.

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# The finance connection

Finance variant:
Given a double continuum of call prices construct a model consistent with those prices.

Traditionally in finance we compute model-based option prices.
In practice, vanilla option prices are fixed by supply and demand.
We want to know:

- is there an arbitrage - if so, how to exploit it
- how to price exotics

Here, vanilla = puts and calls, but might also include barriers, variance swaps, . . .

# Objectives

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# Literature

Single marginal version: the Skorokhod embedding problem;
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Non-trivial starting laws:
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Continuum of marginals:
Gyöngy (1986), Dupire (1997), Madan-Yor (2002), Hirsch-Profeta-Roynette-Yor (2011), Henry-Labordère-Tan-Touzi (2014)

Fake Brownian motion:
Hamza-Klebaner (2002), Oleszkiewicz (2008), Albin (2008)

# A new class of fake diffusions

# Notation and Assumptions

Notation/assumptions: suppose we are given $(\mu_t)_{t\geq 0}$ satisfying

1. Convex Order Assumption
   (a) $\int |y|\mu_t(dt) < \infty$; $\int y\mu_t(dy) = 0\ \forall t$.
   (b) $C(t,x) := \int (y-x)^+\mu_t(dy)$ is increasing in $t$ for each $x$.
   Note, by definition $C(t,x)$ is convex in $x$.
2. Smoothness in Time $Q(t,x) = \dot{C}(t,x)$ exists.
3. Representation of $Q$ as a signed measure Suppose for each $t$, $\chi_t(dx) = (Q_t'')^-(dx)$ and $\nu_t(dx) = (Q_t'')^+(dx)$ are finite, mutually singular measures with the same mass and mean.
4. Finite activity $\chi_t(\mathbb{R})$ is bounded on compact sub-intervals of $(0,\infty)$.

## The regular case

In nice cases we have that $\mu_t$ has a density $\rho_t = \rho(t, \cdot)$ so that $\mu_t(dx) = \rho(t, x)dx$. For the regular case we have the extra assumptions

1. Smoothness in time and space $\dot{\rho}(t, x)$ exists. Then $\chi_t(dx) = (\dot{\rho}(t, x))^- dx$ and $\nu_t(dx) = (\dot{\rho}(t, x))^+ dx$
2. Dispersion Assumption For each fixed $t$, $\dot{\rho}(t, x) < 0$ on an interval $E_t = (f(t), g(t)) \subset \mathbb{R}$ and $\dot{\rho}(t, x)$ is non-negative elsewhere. Moreover, $0 \in E_s \subset E_t$ for $s < t$.
3. Finite activity $q(t, x) := -(\dot{\rho}(t, x)/\rho(t, x))I_{\{x \in E_t\}} \leq K(t)$ for some positive decreasing function $K$.

## Example I: Brownian motion

$$
\begin{aligned}
C(t,x) &= -x\overline{\Phi}\left(\frac{x}{\sqrt{t}}\right) + \sqrt{\frac{t}{2\pi}}e^{-x^2/2t} \\
\rho(t,x) &= \frac{1}{\sqrt{2\pi t}}e^{-x^2/2t} \\
\dot{\rho}(t,x) &= \frac{\rho(t,x)}{2t^2}(x^2 - t) \\
E_t &= (-\sqrt{t}, \sqrt{t}) \\
q(t,x) &= \frac{1}{2t}\left[1 - \frac{x^2}{t}\right] I_{\{-\sqrt{t}<x<\sqrt{t}\}} \\
K(t) &= \frac{1}{2t}
\end{aligned}
$$

# Example II: Continuous Uniform: $\mu_t \sim U[-t,t]$.

$$\begin{aligned} C(t,x) &= \frac{(t-x)^2}{4t} \qquad\qquad -t<x<t \\ \rho(t,x) &= \frac{1}{2t} \qquad\qquad -t<x<t \\ Q(t,x) &= \frac{t^2-x^2}{4t^2} I_{\{-t<x<t\}} \\ \dot\rho(t,x) &= -\frac{1}{2t^2} I_{\{-t<x<t\}} \\ E_t &= (-t,t) \end{aligned}$$

$$q(t,x) = \frac{1}{t} I_{\{-t<x<t\}} \qquad\qquad K(t) = \frac{1}{t}$$

$$\chi_t \sim \frac{1}{t} U[-t,t] \qquad\qquad \nu_t \sim \frac{1}{t} U\{-t,t\}$$

# Examples III: Discrete Uniform: $\mu_t \sim U\{-t, t\}$.

$$
\begin{aligned}
C(t,x) &= \frac{(t-x)}{2} \qquad\qquad -t < x < t \\
Q(t,x) &= \frac{1}{2} I_{\{-t<x<t\}}
\end{aligned}
$$

$Q$ is not the difference of two convex functions.

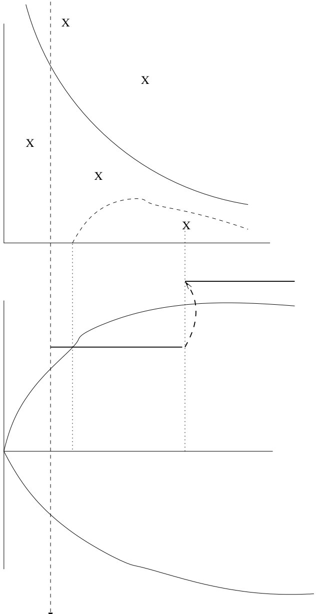
# A Construction under the stronger assumptions

Let $(N_t)_{t \geq 0}$ be a Poisson point process on $(0, \infty) \times (0, \infty) \times (0, 1)$ with density $ds \times dh \times du$.

Suppose that at time $v$ the process is at $x$. Then $X$ stays constant until

$$\tau = \inf\{s > v; \exists (s, h, u) \in \mathcal{N} \cap (h \leq q(x, s))\}$$

Conditional on there being an event of the Poisson Process at $(s, h, u)$ for which $q(x, s) < h$ then the process jumps. If $u < (b(s, x) - x)/(b(s, x) - a(s, x))$ then the process jumps down to $a(s, x)$, otherwise it jumps up to $b(s, x)$.


x
x
x
x
x

# Post-jump locations

Recall that $Q(t,x) = \dot{C}(t,x) \geq 0$
and that for fixed $t$, $Q''(t,x) = \dot{\rho}(t,x) \leq 0$ on $E_t$
and $Q''(t,x) = \dot{\rho}(t,x) \geq 0$ on $E_t^c$.

For $x \in E_t$ and $y > x$ define

$$\mathcal{Q}_{x,t}(y) = Q(t,x) + Q'(t,x)(y-x) - Q(t,y).$$

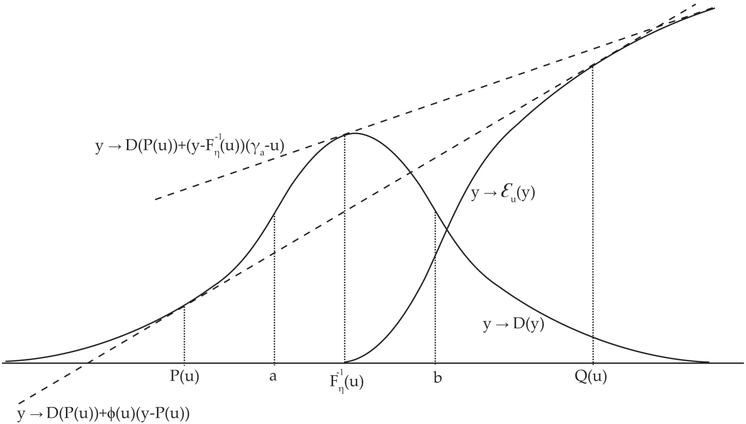
Let $a = a(x,t)$ and $b = b(x,t)$ solve

$$\mathrm{argmax}_{\alpha < x < \beta} \left\{ \frac{\mathcal{Q}_{x,t}(\beta) - Q(t,\alpha)}{\beta - \alpha} \right\}.$$

## Proposition

*For $x \in E_t$, $a$ is differentiable in $x$ and satisfies*

$$a'(t,x)\dot{\rho}(t,a(t,x)) = \frac{b(x,t) - x}{b(x,t) - a(x,t)} \dot{\rho}(t,x)$$



y → D(P(u))+(y-F_η^{-1}(u))(γ_a-u)
y → ℰ_u(y)
y → D(y)
P(u)
a
F_η^{-1}(u)
b
Q(u)
y → D(P(u))+ϕ(u)(y-P(u))

## Theorem

*$X_t$ has law $\mu_t$*

Let $H$ be a test function with support in $E_t$. Then (recall $q = -\dot{\rho}/\rho$ on $E_t$

$$\frac{d}{dt}\mathbb{E}[H(X_t)] = -\int_{E_t} dx \rho(t,x) q(t,x) H(x) = \int_{E_t} H(x)\dot{\rho}(t,x)dx$$

Now let $H$ be a test function with bounded support contained in $(-\infty, f(t))$. Then,

$$\begin{aligned}\frac{d}{dt}\mathbb{E}[H(X_t)] &= \int_{E_t} dy \rho(t,y) q(t,y) \frac{b(t,y)-y}{b(t,y)-a(t,y)} H(a(y,t)) \\ &= -\int_{E_t} H(a(y,t)) a'(t,y) \dot{\rho}(t, a(t,y)) dy \\ &= \int_{\{x \le f(t)\}} H(x)\dot{\rho}(t,x)dx\end{aligned}$$

# A construction under the more general assumptions

Recall, for each $t$, $\chi_t(dx) = (Q_t'')^-(dx) = (\dot\rho(t,x))^- dx$ and $\nu_t(dx) = (Q_t'')^+(dx) = (\dot\rho(t,x))^+ dx$ are finite, mutually singular measures with the same mass and mean.

Let $(N_t)_{t\geq 0}$ be a Poisson point process on $(0,\infty)\times(0,\infty)\times\mathcal{S}$ where $\mathcal{S}$ is a space of realisations of a time inhomogeneous Poisson process. The density of the first two coordinates is $dt \times dk$.

Suppose that at time $v$ the process is at $x$. $X$ stays constant until $\tau = \inf\{t > v; \exists (t,h) \in \mathcal{N} \cap (h \leq \chi_t(\mathbb{R})\}$.

Conditional on their being an event of the Poisson point process at $(t,h)$ for which $h \leq \chi_t(\mathbb{R})$ then the process jumps.

We need to describe a mechanism for those jumps such that the pre-jump value $X$ has law $\chi_t$ and the post-jump value $Y$ has law $\nu_t$ and the martingale property is respected.

Given measures $\chi = \chi_t$ and $\nu = \nu_t$. Define

$$\begin{aligned} Q(x) &= C_\nu(x) - C_\chi(x) \qquad \text{Note } Q(x) = Q(t,x) \\ g(z) &= \arg\inf \frac{Q(x) - Q(y)}{x - y} \\ \gamma(x) &= \inf_{y<x} \frac{Q(x) - Q(y)}{x - y} \\ \Gamma(x) &= \nu(-\infty, x) - \gamma(x) \end{aligned}$$

$\Gamma$ is increasing and plays the role of a left-continuous distribution function.

Let $\chi$ and $\nu$ be measures with the same mass and mean. Suppose $\chi \leq \nu$ in convex order.

Let $X \sim \chi$. Write $F_\chi(x) = \chi((-\infty, x])$ even though $\chi$ need not be a probability measure. Similarly for $F_\nu$.

Let $P_1$ be a Poisson process with rate $m(dz)$ where

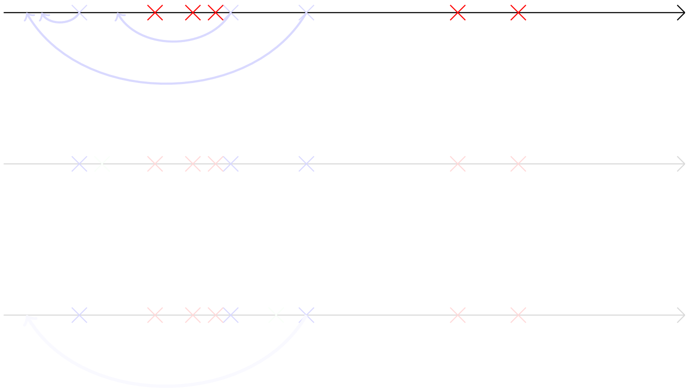
$$m([a,b)) = \int_{a,b} \frac{F_\nu^c(du)}{F_\chi(u-) - \Gamma(u)} + \sum_{u \in [a,b)} \frac{\Delta F_\nu(u)}{F_\chi(u-) - \Gamma(u)}$$

Let $P_2$ be a Poisson process with rate $dz/(z - g(z))$.

Let $S$ be the first event of the Poisson process $P = P_1 + P_2$ which is greater than $X$.

If this event is an event of $P_1$, let $Y = S$. If this event is an event of $P_2$, let $Y = g(S)$.

$P_1$ and $P_2$ are the component parts of the inhomogeneous Poisson Process $N$:



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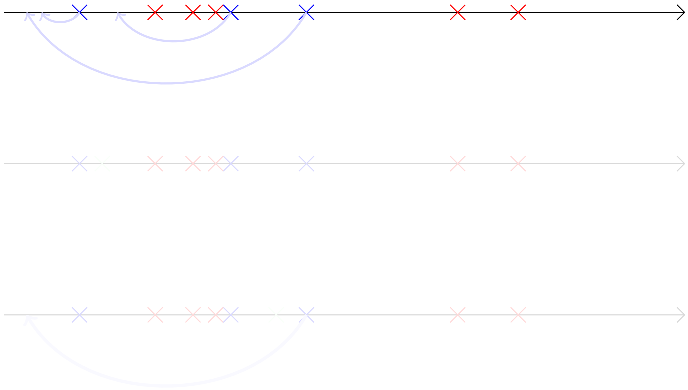

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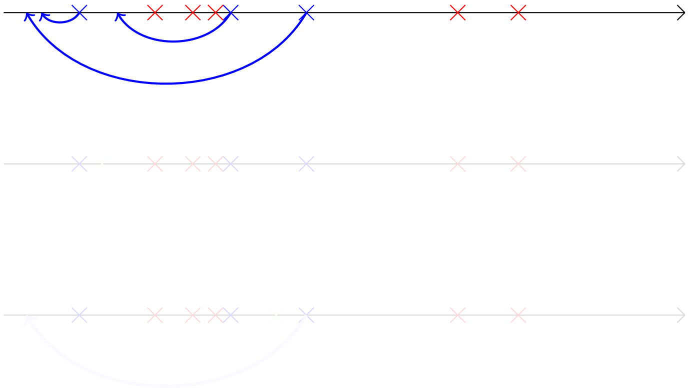

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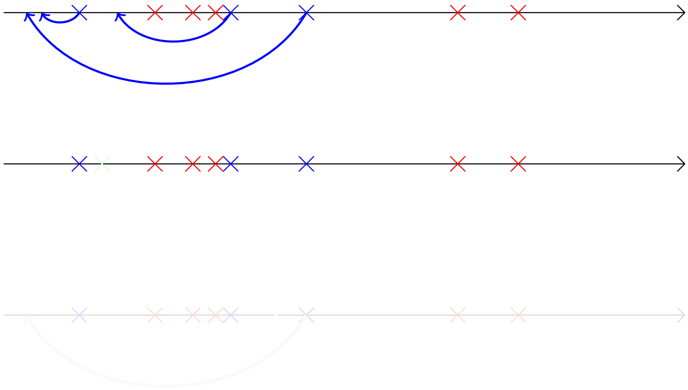

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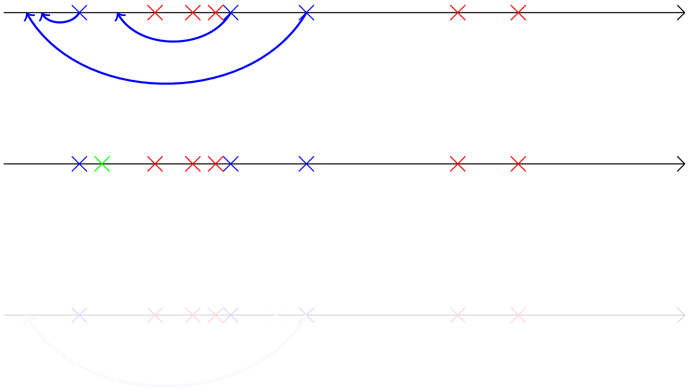

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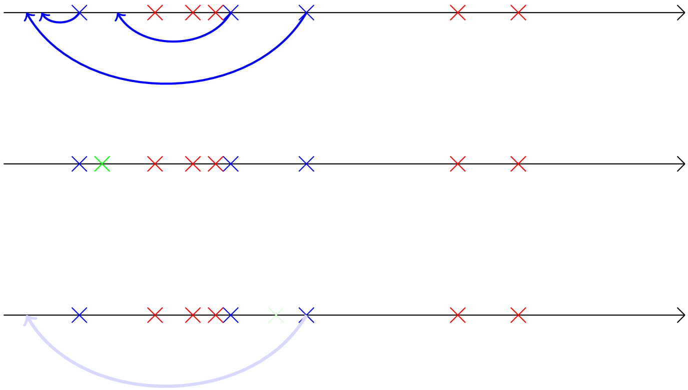

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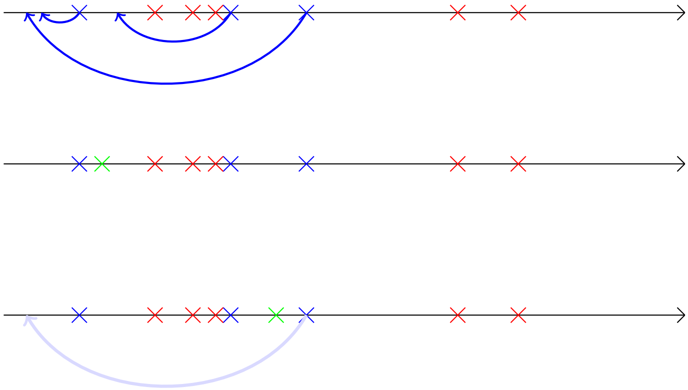

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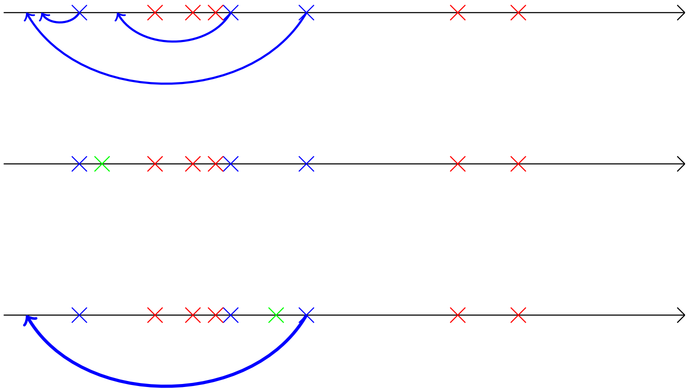

## Theorem

*Let $\zeta = \zeta(dx, ds, dy)$ be the law of $(X, S, Y)$. Then the marginals of $(X, S, Y)$ are such that $X \sim \chi$, $Y \sim \nu$ and $\eta(S < s) = \Gamma(s)$. Moreover $\int_y \int_s (y - x)\zeta(dx, dy, ds) = 0$.*

# Example

$\chi \sim \beta U[-1,1] \qquad \nu \sim \frac{\alpha\beta}{2} U[-n,-1] + \beta(1-\alpha)\delta_0 + \frac{\alpha\beta}{2} U[1,n].$

For $\chi \leq \nu$ in convex order we need $\alpha(n+1) \geq 1$.

$$Q(x) = \begin{cases} 0 & |x| \geq n \\ \frac{\beta\alpha}{4(n-1)}(n-|x|)^2 & |x| \in [1,n) \\ \beta\left(\alpha\left[\frac{(n+1)}{4} - \frac{|x|}{2}\right] - \frac{(1-|x|)^2}{4}\right) & |x| < 1 \end{cases}$$

There are two cases. We assume $\alpha \geq \alpha^*$ where $\alpha^*$ solves $(1-2\alpha)\sqrt{n} + \sqrt{\alpha} = 0$. Let $\eta = \sqrt{1 + (n-1)/\alpha}$.

Then

$$g(x) = \begin{cases} x & -n \leq x \leq -1 \\ x - \eta(1+x) & -1 <\leq x \leq 0 \\ x - \sqrt{4xn + \eta^2(1-x)^2} & 0 < x \leq 1 \\ x - 2\sqrt{xn} & 1 < x \leq n \end{cases}$$

Then for $-1 < x < z < 0$

$$\int_x^z \frac{dv}{v - g(v)} = \int_x^z \frac{dv}{\eta(1+v)} = \frac{1}{\eta} \ln \frac{(1+z)}{(1+x)}$$

$$\exp\left(-\int_x^z \frac{dv}{v - g(v)}\right) = \left(\frac{(1+x)}{(1+z)}\right)^{1/\eta}$$

$$\int_{-1}^0 \chi(dx) \exp\left(-\int_x^0 \frac{dv}{v - g(v)}\right) = \frac{\beta}{2(1+1/\eta)}$$

Recall the rate of the Poisson process $P_1$:

$$m([a,b)) = \int_{a,b} \frac{F_\nu^c(du)}{F_\chi(u-) - \Gamma(u)} + \sum_{u \in [a,b)} \frac{\Delta F_\nu(u)}{F_\chi(u-) - \Gamma(u)}$$

There is no mass on $(x, 0)$, but there is a mass at zero of size $(1-\alpha)/\{\eta/(2(1+\eta))\} = 2(1+\eta)(1-\alpha)/\eta$.

$P_2$ is a Poisson process with rate $dz/(z-g(z))$. Hence the chance that $X < 0, S \geq 0$ is given by

$$\rho(X < 0, S \geq 0) = \frac{\beta}{2(1+1/\eta)}$$

and $\rho(Y = 0) = \rho(X < 0, S = 0)$ where we calculate

$$\rho(X < 0, S = 0) = \frac{\beta}{2(1+1/\eta)} 2 \frac{(1+\eta)}{\eta}(1-\alpha) = \beta(1-\alpha) = \nu(\{0\})$$

# Minimising the 1-variation

Fix $t$. Define $\theta = \theta_{t,a,b} : [f(t), g(t)] \to \mathbb{R}$ and $\alpha = \alpha_{t,a,b} : [f(t), g(t)] \to \mathbb{R}$ via

$$\begin{aligned} \theta(x) &= \int_0^x \frac{2dz}{b(t,z) - a(t,z)}, \\ \alpha(x) &= \int_0^x \frac{2x - a(t,z) - b(t,z)}{b(t,z) - a(t,z)} dz \end{aligned}$$

Extend these definitions to $\mathbb{R}$ by defining $\delta_t = \delta_{t,a,b} : \mathbb{R} \to \mathbb{R}$ and $\psi_t = \psi_{t,a,b} : \mathbb{R} \to \mathbb{R}$ via

$$\delta_t(x) = \begin{cases} \theta(a_t^{-1}(x)) & x < f(t), \\ \theta(x) & x \in E_t, \\ \theta(b_t^{-1}(x)) & x > g(t). \end{cases}$$

$$\psi_t(x) = \begin{cases} \alpha(a_t^{-1}(x)) + (a_t^{-1}(x) - x)(1 - \theta(a_t^{-1}(x))) & x < f(t) \\ \alpha(x) & x \in E_t, \\ \alpha(b_t^{-1}(x)) + (b_t^{-1}(x) - x)(-1 - \theta(b_t^{-1}(x))) & x > g(t). \end{cases}$$

Set $L(x,y) = |y-x| + \psi_t(x) + \delta_t(x)(y-x) - \psi_t(y)$.

**Theorem (Hobson-Klimmek Theorem 4.5)**

*$L(x,y) \geq 0$, with equality for $y \in \{a(t,x), x, b(t,x)\}$.*

Hence $\psi_t(y) \leq |y-x| + \psi_t(x) + \delta_t(x)(y-x)$, and

$$|y-x| \geq \psi_t(y) - \psi_t(x) - \delta_t(x)(y-x)$$

$$\begin{aligned}
&\sum_{k=1}^{n} |y_{t_k} - y_{t_{k-1}}| \\
&\quad \geq \sum_{k=1}^{n} \left(\psi_{t_{k-1}}(y_{t_k}) - \psi_{t_{k-1}}(y_{t_{k-1}})\right) - \sum_{k=1}^{n} \delta_{t_{k-1}}(y_{t_{k-1}})(y_{t_k} - y_{t_{k-1}})
\end{aligned}$$

# Total variation down a nested sequence of partitions

Let $\mathcal{P}_n^{\epsilon,T} = \{t_0^n, t_1^n, \dots t_{N(n)}^n\}$ be a partition of $[\epsilon, T]$.

Let $\mathcal{P}^{\epsilon,T} = (\mathcal{P}_n^{\epsilon,T})_{n \geq 1}$ be a dense sequence of nested partitions.

Let $Y$ be any process with marginals $(\mu_t)_{\epsilon \leq t \leq T}$.

Define the total variation

$$\mathcal{V}_n^{\epsilon,T}(Y) = \mathcal{V}(\mathcal{P}_n^{\epsilon,T}, Y)(\omega) = \sum_{k=1}^{N(n)} |Y_{t_k^n}(\omega) - Y_{t_{k-1}^n}(\omega)|$$

Let $V_n^{\epsilon,T}(Y) = \mathbb{E}[\mathcal{V}_n^{\epsilon,T}(Y)]$ and $V_\infty^{\epsilon,T}(Y) = \lim_n V_n^{\epsilon,T}(Y)$.

## Theorem

$V_{\infty}^{\epsilon,T}(Y) \geq \int_{\epsilon}^{T} dt \int dx \psi_t(x)\dot{\rho}(t,x)$

$$
\begin{aligned}
V_n^{\epsilon,T}(Y) &\geq \mathbb{E}\left[\sum_{k=1}^{N(n)} \psi_{t_{k-1}}(Y_{t_k}) - \psi_{t_{k-1}}(Y_{t_{k-1}})\right] \\
&= \sum_{k=1}^{N(n)} \int dx \psi_{t_{k-1}}(x)\left[\rho(t_{k-1},x) - \rho(t_k,x)\right] \\
&= \sum_{k=1}^{N(n)} \int dx \psi_{t_{k-1}}(x) \int_{t_{k-1}}^{t_k} \dot{\rho}(t,x) dt \\
&\to \int dx \int_{\epsilon}^{T} dt \psi_t(x)\dot{\rho}(t,x)
\end{aligned}
$$

## Concluding Remarks

Previously study has concentrated on constructing extremal models, and associated robust bounds given prices of co-maturing vanilla options.

Even there there is scope for incorporating more exotic options (variance derivatives, barriers on FX markets) as vanilla instruments.

There are many problems of interest which involve incorporating option prices on intermediate dates.

Here we take this to its logical extreme and imagine a full continuum of marginals.

Constructions for special cases (elegant or otherwise) are very valuable, as are approaches which work in full generality.

# Literature

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